Homework 1 – Solutions

① **Claim:** There are infinitely many primes.

Pf: Assume by contradiction that there are only finitely many primes, which are $p_1, p_2, \dots, p_n$.

Now construct a number P such that

$$P = p_1 p_2 p_3 \cdots p_n + 1$$

P has remainder 1 when divided by the n primes, thus not divisible by these n prime numbers. This makes P itself prime $\Rightarrow\Leftarrow$

Therefore, there must be infinitely many primes. ∎

② **Claim:** $\lim_{x \to 4} x^2 = 16$

Pf: Given $\varepsilon > 0$. Let $\delta = \min\{\frac{\varepsilon}{9}, 1\}$.

If $0 < |x-4| < \delta$, then $3 < x < 5$. This implies that $7 < x+4 < 9$.

Thus
$$\begin{aligned}|x^2 - 16| &= |x-4||x+4| \\ &< \frac{\varepsilon}{9} \cdot 9 \\ &= \varepsilon\end{aligned}$$

So $\lim_{x \to 4} x^2 = 16$. ∎

③ **Claim:** $\lim_{x \to \infty} \frac{1}{x^r} = 0$

Pf: Given $\varepsilon > 0$, let $M = \varepsilon^{-1/r}$. If $x > M$, we have

$$\left|\frac{1}{x^r} - 0\right| = \left|\frac{1}{x^r}\right| = \frac{1}{x^r} < \frac{1}{M^r} = \varepsilon$$

Thus $\lim_{x \to \infty} \frac{1}{x^r} = 0$. ∎

④ $$\lim_{x\to\infty} \frac{5x^3-17}{11x^3+2x^2-4x+1}$$

for $x>0$, then $x^3>0$, so

$$\lim_{x\to\infty} \frac{5x^3-17}{11x^3+2x^2-4x+1} = \lim_{x\to\infty} \frac{\frac{5x^3-17}{x^3}}{\frac{11x^3+2x^2-4x+1}{x^3}}$$

$$= \lim_{x\to\infty} \frac{5-\frac{17}{x^3}}{11-\frac{2}{x}-\frac{4}{x^2}+\frac{1}{x^3}} = \boxed{\frac{5}{11}}$$

⑤ $$\lim_{x\to\infty} \frac{x}{\lfloor x \rfloor}$$

Notice $x-1 \le \lfloor x \rfloor \le x$ for all x.

then $$\frac{x}{x} \le \frac{x}{\lfloor x \rfloor} \le \frac{x}{x-1}$$ for all positive x.

$$1 = \lim_{x\to\infty} \frac{x}{x} \le \lim_{x\to\infty} \frac{x}{\lfloor x \rfloor} \le \lim_{x\to\infty} \frac{x}{x-1} = 1$$

then by Squeeze Thm, $\lim_{x\to\infty} \frac{x}{\lfloor x \rfloor} = \boxed{1}$